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Singular Solutions of Ordinary Differential Equations.

BY HENRY B. FINE.

The method best suited to a general investigation of the conditions of occurrence of singular solutions of differential equations and of the properties of these solutions would seem to be that introduced by Briot and Bouquet in their study of ordinary solutions, and developed very beautifully in their memoir, *Propriétés des Fonctions définies par des Equations Différentielles.*[*]

By this method the theory of singular solutions may be based immediately on the differential equation, and any use, direct or indirect, of the notion of the complete primitive be avoided; it is not required to restrict the variables or coefficients of the equation to real values, and account may be taken of the region, not only of points which are ordinary points for the equation, but of such as are its singular points[†] as well.

There are obvious objections to the use of the notion of a complete primitive in any investigation of singular solutions which seeks to be general or rigorous. In the first place it is an inversion of the natural order of investigation, since what is sought is the conditions which a differential equation $f(x, y, p) = 0$ (or one of higher order) must fulfil in order that a curve not properly contained in the system which it defines may satisfy it; not the characteristics of a system of curves defined by some equation $\phi(x, y, \alpha) = 0$ between x, y, and a parameter α.

But in the second place, while the curve system $\phi(x, y, \alpha) = 0$ generally has an envelope, and therefore the differential equation $F(x, y, p) = 0$ by which it may also be defined, a singular solution—the *general* differential equation $f(x, y, p) = 0$ has no singular solution—the curve system which it defines, no envelope. If the equation be of the n^{th} degree, there will pass through any ordinary point x_0, y_0 n curves which satisfy it; but *all* that it gives to define each of these curves is a development of y in positive powers of x. The coefficients in any of these developments are, it is true, rational functions of x_0, y_0 and of one or other of the roots, p_0, of $f(x_0, y_0, p) = 0$, and therefore algebraic functions of a single parameter. But an envelope is not to be had by variation of this parameter; for, at the points of $\text{Disct}_p f(x, y, p) = 0$, the curve which, if any, is the envelope of the system defined by $f(x, y, p) = 0$, the development generally becomes divergent and so ceases altogether to represent a solution of the equation.*

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[*] Journal de l'Ecole Polytechnique, Cah. 36.

[†] Poincaré has given this name to points at which $\frac{dy}{dx}$ is indeterminate; in the case of the equation $\frac{dy}{dx} = \frac{Y}{X}$, for instance, to the points of intersection of $X = 0$, $Y = 0$.

Briot and Bouquet have shown that all ordinary solutions of the equation $f(p, y, x) = 0$ which pass through a given point x_0, y_0, and have there $p = p_0$ in common, may by suitable transformations be reduced to the form $y = Vvx^m$; where V is a determinate function of v and x, and v is connected with x by a differential equation of the form $\frac{dv}{dx} x = \phi(v, x)$. In the first of the following sections it is proven that if $f(p, y, x) = 0$ have a singular solution and x_0, y_0 lie on it, among the functions V determined by the Briot-Bouquet method will be one which corresponds to the singular solution—but that the differential equation between v and x which corresponds to this value of V degenerates into the form

$$\frac{dv}{dx} x = \frac{\psi(v, x)}{\psi(v, x)} \chi(v, x),$$

generally satisfied by $\psi(v, x) = 0$ only. This result brings out clearly the exact relation in which the singular solution stands to the differential equation itself. In the strict sense it is not contained in the equation any more than any arbitrary function $\phi(x, y) = 0$ is contained in the equation $\frac{dy}{dx} = \frac{\phi(x, y)}{\phi(x, y)} \psi(x, y)$. Yet it takes the place of one of the two solutions ordinarily occurring for values x_0, y_0, p_0 of x, y, p, which satisfy the two equations $f = 0$, $\frac{\partial f}{\partial p} = 0$.

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* Darboux was apparently the first to call attention to and explain the paradox involved in the old (Lagrange) theory of singular solutions, according to which, since every differential equation of the first order is to be regarded as having a complete primitive of the form $\phi(x, y, a) = 0$, and $\phi(x, y, a) = 0$ as generally having an envelope, the differential equation should generally have a singular solution. *Vid.* Comptes rendus, t. 70 and 71, and Mémoires de l'Institut, t. XXVII.

In the second and principal section of the paper, the theory developed in the first is extended to equations of higher orders.

Section 3 contains a discussion of the singular solutions of those differential equations of the n^{th} order whose general solution is obtainable in the form $\phi(x, y, \alpha, \beta, \gamma, \ldots . \nu) = 0$, with special reference to the geometrical interpretation of the equation which, algebraically speaking, contains the singular solution. This section is added in deference to a usage which has become almost classical, of regarding singular solutions from the double standpoint of the differential equation itself and its complete primitive. The objection to basing the general theory of singular solutions on that of systems of curves defined by equations of the form $\phi(x, y, \alpha, \beta, \gamma, \ldots . \nu) = 0$ has already been stated; the curve system and the differential equation which represents it may nevertheless be of sufficient interest on their own account to merit the space given them. ϕ, it should be added, is supposed to be a one-valued function of x, y within the region in which it is studied, and a rational function of n parameters $\alpha, \beta, \ldots . \nu$, or, more generally, of m parameters connected by $m - n$ algebraic equations.

§1.

Let $f(x, y, p)$ be any rational integral function of x, y and p, irreducible with respect to p and containing no mere x, y factor: though the discussion which is to follow applies as well to any function of the form $f_0 p^m + f_1 p^{m-1} + \ldots f_m$, if $f_0, f_1, \ldots .$ be holomorphic functions of x and y with a common region of convergence and x, y throughout the discussion be restricted to that region.

The differential equation

$$f(x, y, p) = 0 \tag{1}$$

defines a system of curves, m of which, speaking generally, pass through any point x_0, y_0, and for each of these m curves a development $y - y_0 = \phi(x - x_0)$ is to be had by the following method, first given by Briot and Bouquet.

It is for the sake of convenience and definiteness of statement only that we call x_0, y_0 a point, for complex as well as real values of x and y are taken account of in the discussion.

Let it be assumed that the coefficient of the highest power of p in f does not vanish for $x = x_0$, $y = y_0$: the values of p corresponding to $x_0 y_0$ and given by the algebraic equation $f(x_0, y_0, p) = 0$, are then all finite; call any one of them p_0.

(A method for dealing with the case when the coefficient of the highest power of p vanishes (which Briot and Bouquet do not consider) is given in my paper "On Functions defined by Differential Equations, etc.,"* and is also stated in §2 of the present paper.)

First of all make in $f = 0$ the substitutions

$$\begin{aligned} x &= x_0 + x' \\ y &= y_0 + p_0 x' + y' \\ p &= p_0 + p', \end{aligned} \tag{2}$$

and develop by Taylor's theorem. We then have

$$0 = F(x', y', p') \equiv \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}\right)_0 x' + \left(\frac{\partial f}{\partial y}\right)_0 y' + \left(\frac{\partial f}{\partial p}\right)_0 p' + \dots \tag{3}$$

where p' as well as y' vanishes with x'.

1. Suppose $\left(\frac{\partial f}{\partial p}\right)_0$ and $\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}\right)_0$ to be both different from zero.

Equation (3) then gives for p' a single development in integral powers of x', y', call it $\phi(x', y')$; and the equation $\frac{dy'}{dx'} = \phi(x', y')$ defines y' as a holomorphic function of x',† which is the only solution of (3) vanishing with x' in this the case of usual occurrence.

2. Let $\left(\frac{\partial f}{\partial p}\right)_0 = 0$, but $\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial y}\right)_0 \neq 0$.

In this case p_0 is a repeated root of the equation $f(x_0, y_0, p) = 0$. To have the general case of repeated roots before us, let the multiplicity of the root p_0 be supposed to be r; then $\left(\frac{\partial f}{\partial p}\right)_0 = \left(\frac{\partial^2 f}{\partial p^2}\right)_0 = \dots = \left(\frac{\partial^{r-1} f}{\partial p^{r-1}}\right)_0 = 0$, but $\left(\frac{\partial^r f}{\partial p^r}\right)_0 \neq 0$.

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* Amer. Journal XI, 4.

† By Cauchy's theorem. *Vid.* Briot and Bouquet in Journ. de l'Ecole Pol. Cah. 36, and in Théorie des Fonc. Ellipt. p. 325.

The terms of lowest degree in (3) are then $\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)_0 x'$ and $\frac{1}{r!}\left(\frac{\partial^r f}{\partial p^r}\right)_0 p'^r$.

Let $x'=x''^r$. Then (3) gives r developments for p' in integral powers of x'' and y', the first term in each development being one of the values of

$$\left(-\frac{\frac{\partial f_0}{\partial x_0}+\frac{\partial f_0}{\partial y_0}p_0}{\frac{1}{r!}\frac{\partial^r f_0}{\partial p_0^r}}\right)^{\frac{1}{r}} x''.$$

Each of these developments on being substituted for p' in the equation $\frac{dy'}{dx''}=rx''^{r-1}p'$, gives y' as a holomorphic function of x'',* developable, therefore, in integral powers of $x'^{\frac{1}{r}}$.

There are then in this case r solutions connected cyclicly, the lowest power of x' in each development for y' being $x'^{\frac{r+1}{r}}$. Of the n distinct curves defined by $f(x, y, p)=0$ for the region of an ordinary point x_0, y_0, r have been replaced by the cyclicly connected branches of a single curve on which x_0, y_0 is a singular point of the cusp species.

It follows immediately that

Disct$_p$ $f(x, y, p)=0$ *is in the general case a locus of cusps on the curves defined by the equation* $f(x, y, p)=0$.

The possession of cusps is thus shown to be *characteristic* of the curves defined by the general differential equation of the first order when its degree is higher than 1.

It is obvious that when the coefficients of f are real, the curve Disct$_p$ $f=0$ may be used to separate between the regions of real and imaginary solutions of $f=0$; for as the point x_0, y_0 crosses Disct$_p f=0$, two of the corresponding values of p_0 commonly pass from real to conjugate imaginary values, or vice versa. Thus for the equation

$$p^2+x^2+y^2-1=0$$

the circle $x^2+y^2-1=0$ divides the x, y plane into two regions, in one of which the solutions are always real, in the other always imaginary.

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[*] By Cauchy's theorem. *Vid.* foot-note on p. 298.

3. Let $\left(\frac{\partial f}{\partial p}\right)_0 = 0$ and $\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}p\right)_0 = 0$.

There is generally a finite number only of such points, called by Poincaré the "singular points"* of the equation $f = 0$.

Again to have the general case before us, suppose any additional coefficients of F to vanish.

Now since x', y', p' vanish together, any solution of equation (3) may evidently be thrown into the form

$$y' = p'x'v, \tag{4}$$

where v is a function of x' which takes a finite value v_0, different from zero when $x' = 0$.

On substituting this value of y' in (3), that equation goes over into an equation in p', x' with coefficients which are functions of v.

To get the terms of lowest degree in this p', x' equation follow the Puiseux method or any other method applicable to algebraic equations in two variables.

There may be a number of groups of terms of the same degree in the equation, each of which gives for μ (the degree of p' in respect to x') a value such that the terms of the group are the lowest in the equation.

Consider any one such group, for which $\mu = \frac{r}{s}$.

Make the substitutions

$$x' = x''^s, \quad p' = Vx''^r, \quad v = v_0 + v'. \tag{5}$$

V has a finite value different from zero when $x'' = 0$; and v_0, the initial value of v, is equal to $\frac{s}{r+s}$, as may readily be seen by comparing the equations $p' = Vx'^{\frac{r}{s}}$ and $y' = p'x'v$ for $x' = 0$.

The transformed equation—call it $\phi(x'', v', V) = 0$—then gives for V one or more developments in integral powers of v' and x''. It is assumed that the equation $\phi(0, 0, V) = 0$ has no multiple roots.

Let $V = a + bx'' + cv' + \ldots.$ be one of these series.

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[*] *Vid.* Poincaré in Journal de Math. pure et appliquées, III, 7, 8; IV, 1. These singular points will not be confounded with the singular points of Fuchs and other writers on linear differential equations, the points of discontinuity of coefficients of the equation.

The equation connecting v' and x'' is then obtained by making the substitutions (5) in the equation $y' = p'x'v$. It is

$$x''\frac{dv'}{dx''} = -\frac{(r+s)\,Vv' + \left(\frac{s}{r+s} + v'\right)x''\frac{\partial V}{\partial x''}}{V + \left(\frac{s}{r+s} + v'\right)\frac{\partial V}{\partial v'}}\,; \qquad (6)$$

or, replacing V by the series obtained for it,

$$x''\frac{dv'}{dx''} = -\frac{(r+s)^2 a}{(r+s)\,a + sc}\,v' - \frac{sb}{(r+s)\,a + sc}\,x'' + \ldots.$$

Now this is the form to which the solutions of the equation of the first degree, $\frac{dy}{dx} = \frac{Y}{X}$, can be reduced at points where $X = Y = 0$, the "singular points" of this equation. And by the theory of these solutions,* when $-\frac{(r+s)^2 a}{(r+s)\,a + sc}$ is not a positive integer, equation (6) admits always of a monodrome integral vanishing with x''—there being besides an infinite number of non-monodrome integrals when the real part of this coefficient is positive, no other when it is negative.

When, on the other hand, $-\frac{(r+s)^2 a}{(r+s)\,a + sc}$ is a positive integer (6) can be transformed into an equation of the form $t\frac{du}{dt} = u + bt + \ldots$. which has an infinite number of monodrome integrals when $b = 0$, and no monodrome but an infinite number of non-monodrome integrals when $b \neq 0$.

Finally, if $(r+s)\,a + sc = 0$, (6) is of the form

$$x''\frac{dv'}{dx''} = \frac{a_2 v' + b_2 x'' + \ldots.}{a_1 v' + b_1 x'' + \ldots.}$$

which in general has no monodrome integral vanishing with x''. This case, it should be added, presents itself when $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}p$ vanishes for two or any finite number of consecutive points of $\mathrm{Disct}_p f = 0$; that is to say, when singular points of the equation $f = 0$ fall together. The corresponding curve-elements of $\mathrm{Disct}_p f = 0$ satisfy the equation $f = 0$, but the elements which lie to either side of this do not satisfy this equation (*vid.* 4 inf.)

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* *Vid.* Briot and Bouquet in earlier sections of the memoir already referred to, and Poincaré, Journal de Math., III, 7.

To every solution of (6) there of course corresponds one solution of (3) which may be obtained by substituting for v' its value in the equation $y' = Vvx'^{\frac{r+s}{s}}$.

If, on the other hand, V_0 is a multiple root of $\phi(0, 0, V) = 0$, the corresponding series for V obtained from $\phi(x'', v', V) = 0$ may involve fractional powers of x'' and v', and, to render these powers integral, substitutions of the form $x'' = x'''^p$, $v' = v''^q$ are necessary. Equation (6) is replaced by a similar equation between v'' and x''', viz.

$$x''' \frac{dv''}{dx'''} = - \frac{p(r+s)\,Vv''^q + \left(\frac{s}{r+s} + v''^q\right) x''' \frac{\partial V}{\partial x'''}}{q\,Vv''^{q-1} + \left(\frac{s}{r+s} + v''^q\right) \frac{\partial V}{\partial v''}}. \qquad (6')$$

Singular Solutions.

4. Let $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} p = 0$ at every point of $\mathrm{Disct}_p f = 0$, or if this curve be reducible, at every point of at least one of its branches through x_0, y_0.

This case is altogether exceptional; it requires the coefficients of f to be so taken that the eliminant of $f = 0$, $\frac{\partial f}{\partial p} = 0$, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} p = 0$ with respect to any two of the variables x, y, p shall vanish identically. But if it occur,

$\mathrm{Disct}_p f = 0$ satisfies the equation $f = 0$.

First suppose that the surface which the equation $f(x, y, p) = 0$ defines, when x, y, p are regarded as the coordinates of a point in space, has no singularity at the point x_0, y_0, p_0.

Since $\left(\frac{\partial f}{\partial p}\right)_0 = 0$, the equation of the tangent plane to the surface at this point is

$$\left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) = 0.$$

This plane cuts the xy plane in the tangent line to the projection of the intersection of the surfaces $f = 0$ and $\dfrac{\partial f}{\partial p} = 0$. The slope of this tangent line is therefore $-\dfrac{\left(\dfrac{\partial f}{\partial x}\right)_0}{\left(\dfrac{\partial f}{\partial y}\right)_0}$, that is to say, p_0, since $\left(\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} p\right)_0 = 0$.

Thus at each of its points $\text{Disct}_p f = 0$—which is the projection of the intersection of $f = 0$, $\dfrac{\partial f}{\partial p} = 0$—touches a curve of the f-system, and therefore satisfies the equation $f = 0$.

Second, suppose x_0, y_0, p_0 to be a singular point of the surface $f(x, y, p) = 0$. The substitutions (2) which transformed f into F also transform $\dfrac{\partial f}{\partial p}$ into $\dfrac{\partial F}{\partial p'}$ and $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} p$ into $\dfrac{\partial F}{\partial x'} + \dfrac{\partial F}{\partial y'} p'$.

Again, by virtue of the substitutions (4) and (5) we have the identity

$$F(x', y', p') \equiv X \phi(x'', v', V),$$

where X is an irrelevant factor of the form $x''^p v^q V^t$.

Differentiating with respect to v, V and x'' we obtain

$$\frac{\partial F}{\partial y'} V x''^{r+s} = X \frac{\partial \phi}{\partial v} + \frac{dX}{\partial v} \phi,$$

$$\frac{\partial F}{\partial y'} v x''^{r+s} + \frac{\partial F}{\partial p'} x''^{r} = X \frac{\partial \phi}{\partial V} + \frac{\partial X}{\partial V} \phi,$$

$$(r+s) \frac{\partial F}{\partial y'} V v x''^{r+s-1} + r \frac{\partial F}{\partial p'} V x''^{r-1} + s \frac{\partial F}{\partial x''} x''^{s-1} = X \frac{\partial \phi}{\partial x''} + \frac{\partial X}{\partial x''} \phi;$$

whence

$$\left.\begin{aligned} \frac{\partial F}{\partial p'} &= \frac{X}{V x''^r}\left(\frac{\partial \phi}{\partial V} V - \frac{\partial \phi}{\partial v} v\right) + X_1 \\ \frac{\partial F}{\partial x'} + \frac{\partial F}{\partial y'} p' &= \frac{X}{x''^s}\left(\frac{x''}{s}\frac{\partial \phi}{\partial x''} + \frac{\partial \phi}{\partial v}(1 - v) - \frac{r}{s}\frac{\partial \phi}{\partial V} V\right) + X_2 \end{aligned}\right\} \qquad (7)$$

where X_1 and X_2 are terms of the form $M\phi$ and therefore vanish with ϕ.

When, therefore, F, $\dfrac{\partial F}{\partial p'}$ and $\dfrac{\partial F}{\partial x'} + \dfrac{\partial F}{\partial y'} p'$ vanish together, and that not merely at the point $x' = y' = p' = 0$, but all along a common curve through that point, then along this same curve

$$\begin{aligned} &\phi &&= 0, \\ &\frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v &&= 0, \\ &\frac{x''}{s}\frac{\partial\phi}{\partial x''} + \frac{\partial\phi}{\partial v}(1-v) - \frac{r}{s}\frac{\partial\phi}{\partial V} V &&= 0. \end{aligned}$$

In this case, among the roots, $V = a + bx'' + cv' + \ldots$, of $\phi(x'', v', V) = 0$—each of which ordinarily leads, when substituted in (6), to a solution of the equation $F = 0$ (see 3)—is one which gives rise to the same development for v' in powers of x'', whether substituted in $\frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v = 0$, or in

$$\frac{x''}{s}\frac{\partial\phi}{\partial x''} + \frac{\partial\phi}{\partial v}(1-v) - \frac{r}{s}\frac{\partial\phi}{\partial V} V = 0.$$

Call this development $v' = \psi(x'')$. It is, by the method of its derivation $\left(\text{from } \phi = 0 \text{ and } \frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v = 0\right)$, the development which defines a branch of the curve $\text{Disct}_p f = 0$ passing through the point $x' = y' = p' = 0$.

But it also satisfies, in the algebraic sense, the equation (6).

For since $\phi = 0$,

$$\frac{\partial\phi}{\partial x''} dx'' + \frac{\partial\phi}{\partial v} dv + \frac{\partial\phi}{\partial V} dV = 0;$$

whence

$$\frac{\partial V}{\partial x''} = -\frac{\frac{\partial\phi}{\partial x''}}{\frac{\partial\phi}{\partial V}} \text{ and } \frac{\partial V}{\partial v'} = -\frac{\frac{\partial\phi}{\partial v'}}{\frac{\partial\phi}{\partial V}}.$$

So that equation (6) may be thrown into the form:

$$\begin{aligned} x''\frac{dv'}{dx''}\left(\frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v\right) &= -(r+s)\frac{\partial\phi}{\partial V} Vv' + \frac{\partial\phi}{\partial x''} vx'' \\ &= s(1-v)\left(\frac{\partial\phi}{\partial V} V - \frac{\partial\phi}{\partial v} v\right) + v\left(\frac{\partial\phi}{\partial x''} x'' + s\frac{\partial\phi}{\partial v}(1-v) - r\frac{\partial\phi}{\partial V} V\right). \qquad (8) \end{aligned}$$

For the root V of $\phi = 0$ under consideration, therefore—the root which gives rise to the development $v' = \psi(x'')$—the equation (6) is satisfied; not, however, as a differential equation, but through the vanishing of factors algebraic in x'', v', V which appear in all its terms.

Equation (8) brings out very clearly the relation in which the discriminant curve stands to the differential equation, when it satisfies this equation. It is no more a part of the solution of the equation, strictly speaking, than the arbitrary function χ is of the equation $\frac{dy}{dx}\chi = \Phi\chi$. Yet it takes the place of one of the f-curves through x_0, y_0 to which the method of 3 generally leads. For to it there corresponds a root (V) of $\phi = 0$, though all the roots of this equation usually yield proper solutions of $f = 0$, and the equation (6) to which this root leads has no other solution vanishing with x'' than $v' = \psi(x'')$.

The discriminant curve may at particular points have contact of an order higher than the first with curves of the system $f = 0$. The corresponding V_0 is then a multiple root of $\phi(0, 0, V) = 0$, and one is led to equations (6′) between v'' and x''' instead of equations (6) between v' and x''. A mere repetition of the argument of the present section shows, however, that the v'' belonging to the discriminant curve satisfies its equation again in the algebraic sense only.

It is of course possible for the discriminant curve to be at the same time a curve of the system $f = 0$. V_0 is then again a multiple root of $\phi(0, 0, V) = 0$; and there will be two developments for V which, with their corresponding equations (6′) and $y' = Vvx'^{\frac{r+s}{s}}$, define the same curve—once as the discriminant curve and again as a curve of the system $f = 0$.

To suppose of a differential equation that it is satisfied by the discriminant curve, or a branch of this curve, is the same thing, algebraically speaking, as to suppose that it has an infinite number of singular points. As was noticed at the end of 3, when two or more consecutive points of $\text{Disct}_p f = 0$ are singular points of $f = 0$, the corresponding element of $\text{Disct}_p f = 0$ satisfies $f = 0$, but that element only; if then all points of a branch of $\text{Disct}_p f = 0$ are singular points of $f = 0$, this branch must itself satisfy the equation.

§2.

There is no essential difficulty in extending the preceding theory to equations of higher orders.

Consider the equation

$$f \equiv f_0 y_n^m + f_1 y_n^{m-1} + \dots + f_m = 0, \tag{1}$$

where $f_0, f_1, \dots . f_m$ are holomorphic functions of $x, y, y_1, \dots . y_{n-1}$ within common regions of convergence.

This equation defines an n-ply infinite system of curves of which there are, generally speaking, m for each set of values $x_0, y_0, y_1^0, \dots . y_{n-1}^0$ of $x, y, y_1, \dots . y_{n-1}$, within the common regions of convergence of $f_0, f_1, \dots . f_n$; i. e. m which pass through the point x_0, y_0 and there have with one another contact of the $(n - 1)$th order of which the elements are $y_1^0, y_2^0, \dots . y_{n-1}^0$.

The value of y_n belonging to each of these m curves is ordinarily to be obtained directly by solving (1) which is algebraic in y_n, and a development for the curve itself—of y in powers of x—is then to be had by first making in (1) the substitutions,

$$\left.\begin{aligned} x &= x_0 + x', \\ y &= y_0 + y_1^0 x' + y_2^0 \frac{x'^2}{2!} + \dots . y_n^0 \frac{x'^n}{n!} + y' \end{aligned}\right\} \tag{2}$$

and so transforming it into an equation in $x', y', y_1', \dots . y_n'$ in which y' and *all* the differential coefficients $y_1', y_2', \dots . y_n'$ vanish with x': viz. the equation

$$0 = F \equiv \left(\frac{\partial f}{\partial y_{n-1}} y_n + \frac{\partial f}{\partial y_{n-2}} y_{n-1} + \dots . \frac{\partial f}{\partial y} y_1 + \frac{\partial f}{\partial x}\right)_0 x' + \left(\frac{\partial f}{\partial y}\right)_0 y' + \left(\frac{\partial f}{\partial y_1}\right)_0 y_1' + \dots . + \left(\frac{\partial f}{\partial y_n}\right)_0 y_n' + \dots ., \tag{3}$$

and then from this equation deriving a development of y' in powers of x' by the method given in my paper already referred to—"On Functions defined by Differential Equations, etc." and also presented briefly later on in the present paper.

If, however, f_0, the coefficient of y_n^m in (1) vanishes, one or more of the values of y_n^0 are either infinite or cannot be determined by the method just used.

In this case transform (1) by the substitutions:

$$\left.\begin{aligned} x &= x_0 + x', \\ y &= y_0 + y_1^0 x' + y_2^0 \frac{x'^2}{2!} + \dots . + y_{n-1}^0 \frac{x'^{n-1}}{(n-1)!} + y' \end{aligned}\right\} \tag{4}$$

into an equation

$$\psi(x', y', \dots . y_{n-1}', y_n') = 0$$

in which y', y'_1, ... y'_{n-1} vanish with x'; and then, regarding $\psi = 0$ as an equation in x', y', ... y'_{n-1}, y'_n, or in x', y', ... y'_{n-1}, $\frac{1}{y'_n}$ according as f_m also does or does not vanish for $x = x_0$, $y = y_0$, ... $y_{n-1} = y^0_{n-1}$, determine the degree of y' in respect to x' by the polygon construction given in the paper just mentioned.*

Inasmuch as not only f_0 but also other of the coefficients of f may vanish for $x = x_0$, $y = y_0$, ... $y_{n-1} = y^0_{n-1}$, the degree of y' in respect to x'—call it μ—may admit of a number of determinations; that is to say, it may be possible in a number of ways to select groups of terms in $\psi = 0$ which a certain value of μ will render of the same degree with each other, and at the same time of lower degree than the remaining terms in this equation. To each of these values of μ will correspond one or more of the ψ-curves for which y', ... y'_{n-1} vanish with x'.

We consider the cases $\mu > n$, $\mu = n$, $\mu < n$, and in each case transform $\psi = 0$ into an equation in which, as in (3), the n^{th} differential coefficient as well as those of lower orders vanishes with the independent variable.

(a) If $\mu > n$, y'_n vanishes with x' and $\psi = 0$ already has the form $F = 0$. Indeed in this case since $y^0_n = 0$, the substitution (4) is identical with the substitution (2).

(b) If $\mu = n$, set $y' = \frac{vx'^n}{n!}$ in the group of lowest terms in ψ which corresponds to this value of μ, and after removing the common factor x'^n let $x' = 0$.

Each root v_0 of the resulting equation will be the y^0_n of one of the curves sought for, and for each $\psi = 0$ may be transformed into the form $F = 0$ by the substitution

$$y' = \frac{y^0_n}{n!} x'^n + y''. \tag{5}$$

The substitutions (4) and (5) together constitute a substitution (2).

(c) Finally if $\mu < n$, the substitution $x' = x''^m$—where m is the first integer greater than $\frac{n}{\mu}$ — will transform $\psi = 0$ into an equation in which all the differential coefficients to the n^{th} inclusive vanish with x''.

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*I have been at pains to restate the treatment of the case $f_0 = 0$ with some fulness, since the statement of it in my former paper is careless and in one particular incorrect (*vid.* Am. Journ. XI, p. 321, line 18 from top).

Particular mention may be made in passing of the case when for $x = x_0$, $y = y_0, \ldots, y_{n-1} = y^0_{n-1}$, the coefficient f_0 vanishes and that to the first degree in $x - x_0$, $y - y_0, \ldots$, while f_1 takes a value different from zero.

As the polygon construction readily shows, the one group of lowest terms in $\psi = 0$ then consists of the $\frac{1}{y'_n}$ and y'_{n-1} terms, and for this group $\mu = n - \frac{1}{2}$. The theorem* immediately follows:

The differential equation of the $(n-1)^{th}$ order, $f_0 = 0$, defines a system of curves each of which has at every point along it a contact of order $n - 1$ with a curve of the system $f = 0$, which there has a singularity of the cusp class characterized by the development:

$$y = y_0 + y^0_1 x' + y^0_2 \frac{x'^2}{2!} + \ldots + y^0_{n-1} \frac{x'^{n-1}}{(n-1)!} + A x'^{n-\frac{1}{2}} + B x'^n + \ldots$$

In like manner if f_0 and f_1 both vanish to the first degree in $x - x_0$, $y - y_0, \ldots$ but f_2 does not vanish, one value of μ is $n - \frac{1}{3}$; and generally, if $f_0, f_1, \ldots f_{\kappa-1}$ all vanish, but f_κ does not, one value of μ is $n - \frac{1}{\kappa + 2}$.

(It may happen that there are no curves of the system $f = 0$ with the initial elements $x_0, y_0, \ldots y^0_{n-1}$. Thus no curve of the system

$$xy_2 = 1$$

passes through $x = y = 0$ and there has $y_1 = 0$.)

For the sake of definiteness and simplicity of statement, the discussion which follows will be confined to curves of the system for which $f = 0$ can be transformed into the form $F = 0$; that is, to those for which $\mu >$ or $= n$. Of course the curves for which $\mu < n$ admit of a precisely similar treatment; but in consequence of the substitution $x = x'^m$ the coefficients in their developments are differently related to the coefficients of f.

The reduction to the form $F = 0$ having been accomplished, the development for the curve—of y' in powers of x'—may be obtained as follows.

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* This theorem is also given by Goursat, this Journ. XI, 4.

Make in F the substitutions

$$\left.\begin{aligned} y'_{n-1} &= y'_n v_1 x', \\ y'_{n-2} &= y'_n v_1 v_2 x'^2, \\ &\cdots\cdots\cdots\cdots \\ y' &= y'_n v_1 v_2 \ldots\ldots v_n x'^n \end{aligned}\right\} \tag{6}$$

where $v_1, v_2, \ldots\ldots v_n$ are functions of x' which take finite values $v_1^0, v_2^0, \ldots\ldots v_n^0$ different from zero when $x' = 0$.

In the resulting equation, freed from any factor which may be common to all its terms, determine the various groups of terms of lowest order in y'_n and x'. This is, of course, necessary only when certain of the coefficients in (3) vanish, in particular those of y'_n and x'; in which case more than one of the curves or curve-branches of the f-system may have the initial elements $x_0, y_0, \ldots\ldots y_n^0$, and (3) must give rise to a development for each of them.

Suppose that for any particular group of terms of lowest order the degree of y'_n in respect to x' is $\frac{r}{s}$; to obtain the corresponding developments make then the further substitutions

$$x' = x''^s, \quad y'_n = V x''^r, \quad v_\kappa = v_\kappa^0 + v'_\kappa \tag{7}$$

—where again V takes a finite value V_0 different from zero when $x'' = 0$—and thus transform (3) into an equation in $x'', v'_1, v'_2, \ldots\ldots, v'_n, V$; viz.

$$\phi\,(x'', v'_1, v'_2, \ldots\ldots v'_n, V) = 0. \tag{8}$$

The initial value v_κ^0 of v_κ may readily be shown to be $\frac{s}{r + \kappa s}$.

From the equation $\phi = 0$ developments are to be obtained for V in integral powers of x'' and $v'_1, v'_2, \ldots\ldots v'_n$—one for each of the curves the degree of whose y'_n in respect to x' is $\frac{r}{s}$. It is assumed that $\phi\,(0, 0, \ldots\ldots V) = 0$ has simple roots only.*

And finally, differential equations which define the set of values of v'_1,

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* The extension of the entire discussion which follows to the case of multiple roots is too obvious to require special treatment. *Vid.* §1, 3, at end.

$v_2', \ldots . v_n'$ which corresponds to any one of these values of V may be gotten by aid of (6) and (7). They are:

$$\left.\begin{aligned}
& x''\frac{dv_1'}{dx''}\left[v_1\frac{\partial V}{\partial v_1'}+V\right]+x''\frac{dv_2'}{dx''}v_1\frac{\partial V}{\partial v_2'}+\ldots . +x''\frac{dv_n'}{dx''}v_1\frac{\partial V}{\partial v_n'}, \\
& \qquad = -Vv_1'(r+s)-v_1x''\frac{\partial V}{\partial x''}, \\
& x''\frac{dv_2'}{dx''}=\frac{s(v_1-v_2-v_1v_2)}{v_1}, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& x''\frac{dv_n'}{dx''}=\frac{s(v_{n-1}-v_n-v_{n-1}v_n)}{v_{n-1}}.
\end{aligned}\right\} \tag{9}$$

If in the first of these equations there be substituted for the products $x''\frac{dv_2'}{dx''}, x''\frac{dv_3'}{dx''}, \ldots .$ the functions of the v_i''s to which the remaining equations make them equal, the set is reduced to a form from which the quantities $\left(\frac{dv_1'}{dx''}\right)_0, \left(\frac{dv_2'}{dx''}\right)_0, \ldots .$ may be reckoned out by differentiating, setting $x''=0$, and solving the resultant set of linear equations; and from which the values of the higher differential coefficients of the v_i''s with respect to x'' for $x''=0$ may be reckoned out by successive repetitions of this process.

It will be supposed that the determinant of the coefficients in each of these sets of linear equations is different from zero; or (*vid.* this Journ. XI, 324) that

$$\begin{aligned}
& V_0(r+s+k) \quad (r+2s+k)\ldots .(r+ns+k) \\
& +\left(\frac{\partial V}{\partial v_1'}\right)_0\frac{ks}{r+s} \quad (r+2s+k)\ldots .(r+ns+k) \\
& +\left(\frac{\partial V}{\partial v_2'}\right)_0\frac{ks(r+s)}{r+2s}(r+3s+k)\ldots .(r+ns+k) \\
& +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& +\left(\frac{\partial V}{\partial v_n'}\right)_0\frac{ks(r+s)(r+2s)\ldots .(r+(n-1)s)}{r+ns}
\end{aligned}$$

vanishes for no positive integral value of k. The process then leads to determinate series with finite coefficients; viz:

$$v_i'=\left(\frac{dv_i'}{dx''}\right)_0x''+\frac{1}{2!}\left(\frac{d^2v_i'}{dx''^2}\right)_0x''^2+\frac{1}{3!}\left(\frac{d^3v_i'}{dx''^3}\right)_0x''^3+\ldots . \quad (i=1, 2, \ldots . n)$$

which formally satisfy the equations (9) and converge for regions of finite extent.

To obtain the development sought for—of y' in powers of x'—it then only remains to substitute $x'' = x'^{\frac{1}{s}}$ and the series thus obtained for the v_i''s in the equation

$$y' = Vv_1v_2 \dots . v_nx'^n.$$

It will be holomorphic in $x'^{\frac{1}{s}}$ and have the form

$$y' = A_1x'^{n+\frac{1}{s}} + A_2x'^{n+\frac{2}{s}} + A_3x'^{n+\frac{3}{s}} + \dots .$$

The theory thus far developed leads directly to the following conclusions regarding the curves of the f-system which have the initial elements $x_0, y_0, y_1^0, \dots y_n^0$ when for these elements $f = 0$ can be transformed into $F = 0$.

1. If $\left(\frac{\partial f}{\partial y_n}\right)_0 \neq 0$, there is but one group of lowest terms in $F = 0$, hence but one development for V in powers of x' and the variables v_i', and, therefore, speaking generally, but one development for y' in powers of x'; they will be integral powers, of course. In other words, there is but one curve of the system, defined by a differential equation $f(x, y, y_1, \dots . y_n) = 0$, for each set of initial values of the variables, unless besides the equation $f = 0$ itself, these initial values satisfy the equation $\frac{\partial f}{\partial y_n} = 0$.

2. $\left(\frac{\partial f}{\partial y_n}\right)_0 = 0, \left(\frac{\partial f}{\partial y_{n-1}} y_n + \frac{\partial f}{\partial y_{n-2}} y_{n-1} + \dots . + \frac{\partial f}{\partial y} y_1 + \frac{\partial f}{\partial x}\right)_0 \equiv \Delta f_0 \neq 0.$

The group of lowest terms in $F = 0$ then consists of the x' term—of which Δf_0 is the coefficient—and the $y_n'^2$ term; or if also

$$\left(\frac{\partial^2 f}{\partial y_n^2}\right)_0 = \left(\frac{\partial^3 f}{\partial y_n^3}\right)_0 = \dots . \left(\frac{\partial^{s-1} f}{\partial y_n^{s-1}}\right)_0 = 0, \text{ but } \left(\frac{\partial^s f}{\partial y_n^s}\right)_0 \neq 0,$$

of the x' term and the $y_n'^s$ term.

For this group of terms the degree of y'_n in respect to x' is $\frac{1}{s}$; the corresponding V equation $\phi = 0$ (8) has s cyclicly connected roots each of which gives rise finally to a development of the form

$$y' = A_1 x'^{n+\frac{1}{s}} + A_2 x'^{n+\frac{2}{s}} + A_3 x'^{n+\frac{3}{s}} + \dots .$$

In other words, s of the m curves which belong to an ordinary set of initial elements $x_0, y_0, \dots . y^0_{n-1}$ and there have distinct y^0_n's are in the present case replaced by s branches of a single curve which are cyclicly connected and have the same y^0_n. At least two of the m curves will be replaced in this manner for every set of initial elements $x_0, y_0, \dots . y^0_{n-1}$ which satisfy the equation $\mathrm{Disct}_{y_n} f = 0$, got by elimination of y_n from between $f = 0$ and $\frac{\partial f}{\partial y_n} = 0$; that is to say:

The differential equation of the $(n-1)^{th}$ order, $\mathrm{Disct}_{y_n} f = 0$, defines a system of curves each of which has at every one of its points a contact of order $n - 1$ with a curve of the system $f = 0$ which there has a singularity of the cusp class characterized by the development:

$$y = y_0 + y^0_1 x' + y^0_2 \frac{x'^2}{2!} + \dots + y^0_n \frac{x'^n}{n!} + A x'^{n+\frac{1}{2}} + B x'^{n+1} + {}^{*} \dots .$$

For an equation of the first order, as has already been noticed, this singularity is the ordinary cusp; for the equation of the second order, it is the cusp of the second species (Schnabelspitze), etc.

It need hardly be said that $\mathrm{Disct}_{y_n} f = 0$ will in general not satisfy the equation $f = 0$.

3. $\left(\frac{\partial f}{\partial y_n}\right)_0 = 0, \ \Delta f_0 = 0.$

The method of the preceding paragraphs ordinarily gives in this case two developments for y' in *integral* powers of x' both of the form

$$y' = A_1 x'^{n+1} + A_2 x'^{n+2} + A_3 x'^{n+3} + \dots .$$

developments of two curves of the system which have contact of the n^{th} order with each other.

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[*] This theorem is also given by Goursat, this Journ. XI, 4.

A full discussion of all the particular cases which suggest themselves at this point, however,—when other of the coefficients in $F = 0$ besides those of y'_n and x' vanish—would mean the development of the entire theory of singular points (in the Poincaré sense) of differential equations of higher orders—which is quite aside from the purpose of the present paper.

We confine ourselves to the consideration of the hypothesis that Δf vanishes for *all* values of $x, y, \ldots . y_n$ which satisfy both the equations $f = 0$ and $\frac{\partial f}{\partial y_n} = 0$, or—if $\text{Disct}_{y_n} f$ be resolvable into factors—for all values of $x, y, \ldots . y_{n-1}$ with the corresponding y_n—which satisfy the equation got by setting one of these factors equal to zero.

It is of course only in connection with special equations of the n^{th} order that this hypothesis is realized. Ordinarily when $f = 0$, $\frac{\partial f}{\partial y_n} = 0$ and $\Delta f = 0$ together, the initial elements $x, y, \ldots . y_{n-2}$ satisfy an equation of the $(n - 2)^{\text{th}}$ order got by eliminating y_n and y_{n-1} from between $f = 0$, $\frac{\partial f}{\partial y_n} = 0$ and $\Delta f = 0$; but when our hypothesis is realized this eliminant equation vanishes identically—which means, of course, that f has a specialized form.

In this case $\text{Disct}_{y_n} f = 0$ satisfies the equation $f = 0$, but in such manner that the curve system which it defines forms no part of the system defined by $f = 0$. In other words, it is *a singular solution.*

The first statement is capable of a simple geometric demonstration; at least when additional coefficients of F do not vanish, viz:

Regard $x, y, \ldots . y_n$ as point coordinates in a flat space of n dimensions, S_n.

$y_n = 0$ is then the equation of a flat space of $n - 1$ dimensions, S_{n-1}, contained in S_n and in which $x, y, \ldots . y_{n-1}$ are point coordinates.

We thus gain two geometric pictures for the system of curves represented by the equation $f = 0$.

1st. As a system of curves in the curved space $f(x, y, \ldots . y_n) = 0$, one of which passes through each point $x, y, \ldots . y_n$ of this space.

2d. As a system of curves which is the projection of this first system in S_{n-1}, and of which m pass through each point $x, y, \ldots, y_{n-1}$ of this space.

Now the equation of the flat space of $n - 1$ dimensions tangent to $f = 0$ at any point $x_0, \ldots, y^0_n$ where $\frac{\partial f}{\partial y_n} = 0$ is:

$$\left(\frac{\partial f}{\partial y_{n-1}}\right)_0 (y_{n-1} - y^0_{n-1}) + \ldots \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) = 0,$$

which is also the equation of the S_{n-2} in which this tangent intersects S_{n-1}.

But as $x_0, y_0, \ldots y^0_n$ are given all values which satisfy the two equations $f = 0$, $\frac{\partial f}{\partial y_n} = 0$, this S_{n-2} envelopes the curved space of $n - 2$ dimensions which is the picture of $\text{Disct}_{y_n} f = 0$ in S_{n-1}.

It follows immediately that the curves of the system $\text{Disct}_{y_n} f = 0$ satisfy the relation $\left(\frac{\partial f}{\partial y_{n-1}} y_n + \frac{\partial f}{\partial y_{n-2}} y_{n-1} + \ldots \frac{\partial f}{\partial y} y_1 + \frac{\partial f}{\partial x}\right)_0 = 0$.

And since, by hypothesis, the curves of the system $f = 0$ for whose initial elements $x_0, y_0, \ldots y^0_{n-1}$ the equation $\text{Disct}_{y_n} f = 0$ holds good, satisfy this same relation; it is clear that the y^0_n of the curve of $\text{Disct}_{y_n} f = 0$ with the initial elements $x_0, y_0, \ldots y^0_{n-1}$ is the same as the y_n of a curve of $f = 0$ with the same initial elements; or in other words, that the curves of $\text{Disct}_{y_n} f = 0$ satisfy the equation $f = 0$.

Therefore in the case now under consideration,

The differential equation of the $(n - 1)^{th}$ *order,* $\text{Disct}_{y_n} f = 0$, *represents a system of curves each of which has at every one of its points contact of the* n^{th} *order with a curve of the system* $f = 0$.

But the thing most important to consider for its bearing on the theory of singular solutions is the degenerate form which the equations (9) take when Δf vanishes wherever f and $\frac{\partial f}{\partial y_n}$ vanish together. The consideration of this question leads also to the general demonstration that $\text{Disct}_{y_n} f = 0$ then satisfies the equation $f = 0$.

In effecting the transformation of $f = 0$ into $\phi = 0$, f was first transformed into F by means of the substitutions (2), or (4) and (5). It may readily be shown that the same substitutions transform $\frac{\partial f}{\partial y_n}$ into $\frac{\partial F}{\partial y'_n}$ and Δf into ΔF.

Second, by means of the substitutions (6) and (7), $F = 0$ was transformed into $\phi\,(x'', v_1', v_2', \ldots . v_n', V) = 0$. The corresponding transformations of $\frac{\partial F}{\partial y_n'}$ and ΔF are required.

To obtain them set out with the identity

$$F(x', y', \ldots . y_n') = X.\phi\,(x'', v_1', v_2', \ldots . v_n', V),$$

where X is an irrelevant factor of the form $x''^m v_1^{p_1} v_2^{p_2} \ldots . v_n^{p_n} V^q$ which may be common to all the terms of the complete expression in x'', v_1, ... V into which F is transformed by the foregoing substitutions.

Differentiating the two sides of this identity with respect to each of the quantities v_1, ... v_n we get the n identical equations:

$$\begin{aligned}\frac{\partial F}{\partial y'}\,\frac{V v_1 v_2 \ldots . v_n x''^{r+ns}}{v_{n-p}} &+ \frac{\partial F}{\partial y_1'}\,\frac{V v_1 v_2 \ldots . v_{n-1} x''^{r+(n-1)s}}{v_{n-p}} + \ldots . \\ &+ \frac{\partial F}{\partial y_p'}\,\frac{V v_1 v_2 \ldots . v_{n-p} x''^{r+(n-p)s}}{v_{n-p}} \\ &= X\frac{\partial \phi}{\partial v_{n-p}} + \frac{\partial X}{\partial v_{n-p}}\,\phi;\ (p = 0,\ 1,\ 2,\ \ldots . n-1).\end{aligned}$$

And differentiating with respect to V and x'', the additional equations:

$$\frac{\partial F}{\partial y'}\,v_1 v_2 \ldots v_n x''^{r+ns} + \frac{\partial F}{\partial y_1'}\,v_1 v_2 \ldots v_{n-1} x''^{r+(n-1)s} + \ldots + \frac{\partial F}{\partial y_n'}\,x''^{r} = X\frac{\partial \phi}{\partial V} + \frac{\partial X}{\partial V}\,\phi,$$

$$\begin{aligned}\frac{\partial F}{\partial y'}\,(r+ns)\,V v_1 v_2 \ldots . v_n x''^{r+ns-1} &+ \frac{\partial F}{\partial y_1'}\,(r+(n-1)s)\,V v_1 v_2 \ldots . v_{n-1} x''^{r+(n-1)s-1} \\ &+ \ldots . + \frac{\partial F}{\partial y_n'}\,r V x''^{r-1} + \frac{\partial F}{\partial x'}\,s x''^{s-1} = X\frac{\partial \phi}{\partial x''} + \frac{\partial X}{\partial x''}\,\phi.\end{aligned}$$

By solving these equations for $\frac{\partial F}{\partial y'}, \frac{\partial F}{\partial y_1'}, \ldots . \frac{\partial F}{\partial y_n'}, \frac{\partial F}{\partial x''}$ and by a series of reductions which though somewhat tedious present no difficulties, we obtain the expressions which we are seeking, viz:

$$\left.\begin{aligned}\frac{\partial F}{\partial y_n'} &= \frac{X}{V x''^{r}}\left[\frac{\partial \phi}{\partial V}\,V - \frac{\partial \phi}{\partial v_1}\,v_1\right] + Y_1\phi, \\ \Delta F &= \frac{X}{x''^{s}}\left[\frac{x''}{s}\,\frac{\partial \phi}{\partial x''} - \frac{r}{s}\,\frac{\partial \phi}{\partial V}\,V + \frac{\partial \phi}{\partial v_1}(1 - v_1) + \frac{\partial \phi}{\partial v_2}\,\frac{v_1 - v_2 - v_1 v_2}{v_1}\right. \\ &\quad \left. + \frac{\partial \phi}{\partial v}\,\frac{v_2 - v_3 - v_2 v_3}{v_2} + \ldots . + \frac{\partial \phi}{\partial v_n}\,\frac{v_{n-1} - v_n - v_{n-1} v_n}{v_{n-1}}\right] + Y_2\phi.\end{aligned}\right\}(10)$$

Y_1 and Y_2 are linear functions of the irrelevant expressions $\frac{\partial X}{\partial v_i}$, $\frac{\partial X}{\partial V}$, $\frac{\partial X}{\partial x''}$.

It follows at once from (10) that when F, $\frac{\partial F}{\partial y'_n}$ and ΔF vanish together (for $x'' \gtrless 0$ as well as for $x''=0$), the same is true of ϕ, $\frac{\partial \phi}{\partial V}V-\frac{\partial \phi}{\partial v_1}v_1$ and

$$\frac{x''}{s}\frac{\partial\phi}{\partial x''}-\frac{r}{s}\frac{\partial\phi}{\partial V}V+\frac{\partial\phi}{\partial v_1}(1-v_1)+\frac{\partial\phi}{\partial v_2}\frac{v_1-v_2-v_1v_2}{v_1}+\frac{\partial\phi}{\partial v_3}\frac{v_2-v_3-v_2v_3}{v_2}$$
$$+\dots\dots+\frac{\partial\phi}{\partial v_n}\frac{v_{n-1}-v_n-v_{n-1}v_n}{v_{n-1}}.$$

In the light of these results consider the first of the equations of (9).

Since $\phi=0$, $\frac{\partial\phi}{\partial x''}dx''+\frac{\partial\phi}{\partial v_1}dv_1+\frac{\partial\phi}{\partial v_2}dv_2+\dots\dots+\frac{\partial\phi}{\partial v_n}dv_n+\frac{\partial\phi}{\partial V}dV=0$;

whence
$$\frac{\partial V}{\partial v_1}=-\frac{\frac{\partial\phi}{\partial v_1}}{\frac{\partial\phi}{\partial V}},\ \dots\dots\frac{\partial V}{\partial v_n}=-\frac{\frac{\partial\phi}{\partial v_n}}{\frac{\partial\phi}{\partial V}},\ \frac{\partial V}{\partial x''}=-\frac{\frac{\partial\phi}{\partial x''}}{\frac{\partial\phi}{\partial V}}.$$

The use of these equations and the last $(n-1)$ equations of (9) gives the first equation of this set the form:

$$x''\frac{dv'_1}{\partial x''}\left(\frac{\partial\phi}{\partial v_1}v_1-\frac{\partial\phi}{\partial V}V\right)=(r+s)\frac{\partial\phi}{\partial V}Vv_1-\frac{\partial\phi}{\partial x''}v_1x''-sv_1\Big[\frac{\partial\phi}{\partial v_2}\frac{v_1-v_2-v_1v_2}{v_1}$$
$$+\frac{\partial\phi}{\partial v_3}\frac{v_2-v_3-v_2v_3}{v_2}+\dots\dots+\frac{\partial\phi}{\partial v_n}\frac{v_{n-1}-v_n-v_{n-1}v_n}{v_{n-1}}\Big],$$

or when $sv_1\left(-\frac{x''}{s}\frac{\partial\phi}{\partial x''}+\frac{r}{s}\frac{\partial\phi}{\partial V}V-\frac{\partial\phi}{\partial v_1}(1-v_2)\right)$ is added to the bracketed terms of its right number and at the same time subtracted from the terms without the brackets, the form:

$$x''\frac{dv'_1}{\partial x''}\left(\frac{\partial\phi}{\partial v_1}v_1-\frac{\partial\phi}{\partial V}V\right)=s(v_1-1)\left(\frac{\partial\phi}{\partial v_1}v_1-\frac{\partial\phi}{\partial V}V\right)+\Big(\frac{x''}{s}\frac{\partial\phi}{\partial x''}-\frac{r}{s}\frac{\partial\phi}{\partial V}V$$
$$+\frac{\partial\phi}{\partial v_1}(1-v_1)+\frac{\partial\phi}{\partial v_2}\frac{v_1-v_2-v_1v_2}{v_1}+\dots\dots+\frac{\partial\phi}{\partial v_n}\frac{v_{n-1}-v_n-v_{n-1}v_n}{v_{n-1}}\Big).$$

From equations (10) it follows therefore, that in the case before us, this equation regarded as a differential equation is illusory. Values can, it is true, be found for the v_i''s which in an algebraic sense satisfy this and the remaining equations.

From either of the equations $\frac{\partial\phi}{\partial v_1} v_1 - \frac{\partial\phi}{\partial V} V = 0$, or

$$\frac{x''}{s}\frac{\partial\phi}{\partial x''} - \frac{r}{s}\frac{\partial\phi}{\partial V} V + \frac{\partial\phi}{\partial v_1}(1 - v_1) + \frac{\partial\phi}{\partial v_2}\frac{v_1 - v_2 - v_1 v_2}{v_1} + \ldots + \frac{v_{n-1} - v_n - v_{n-1}v_n}{v_{n-1}} = 0$$

a development may be had for v_1' in powers of x'' and v_2', v_3', ... v_n'.

Let this value of v_1' be substituted in the equation

$$x''\frac{dv_2'}{dx''} = \frac{v_1 - v_2 - v_1 v_2}{v_1},$$

when it with the equations still remaining, viz.

$$x''\frac{dv_3'}{dx''} = \frac{v_2 - v_3 - v_2 v_3}{v_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x''\frac{dv_n'}{dx''} = \frac{v_{n-1} - v_n - v_{n-1}v_n}{v_{n-1}},$$

will form a system from which series may be obtained for v_2', v_3', ... v_n' in powers of x'' by the method already described.

The substitution of these values of v_2', v_3', ... v_n' in the development already obtained for v_1' will give a series for it also in powers of x''.

But while these series may in an algebraic sense be called the solution of the equations (9), they are by no means such in the differential equation sense; no more indeed than the arbitrary function defined by the equation $\chi = 0$ is a solution of the differential equation $\frac{dy}{dx}\chi = Y\chi$.

And in the same sense and that only is the series to be gotten for y by the substitution of these values of v'_1, v'_2, ... v'_n in the equation

$$y = y_0 + y_1^0 x' + y_2^0 \frac{x'^2}{2!} + \ldots\ldots + y_n^0 \frac{x'^n}{n!} + V v_1 v_2 \ldots\ldots v_n x'^{n+\frac{r}{s}},$$

a solution of the given equation $f = 0$.

We have thus arrived at a general criterion for singular solutions of the differential equation of any degree, which distinguishes them fully from ordinary or proper solutions and yet shows their place in the system of proper solutions; a criterion not based on geometrical considerations or assumptions as to the character of the curves defined by a differential equation, but based solely and directly on the differential equation itself.

A general equation $f = 0$ will have no singular solution; for it is a necessary condition for the occurrence of such a solution that Δf vanish wherever f and $\frac{\partial f}{\partial y_n}$ vanish together, and this imposes limitations on the coefficients of f.

When a singular solution exists, however, it will be met in applying the general methods which yield the solutions of $f = 0$—usually all proper solutions—which belong to a given set of initial elements x_0, y_0, ... y_n^0. But in its case the first of the set of canonical equations to which $f = 0$ can be reduced for each of these solutions is satisfied not as a differential equation but only through the vanishing of factors algebraic in x'', v'_1, ... v'_n which appear in all its terms.

§3.

Let $$\phi(x, y, c_1, c_2, \ldots\ldots c_m) = 0$$

be any irreducible function of x, y, c_1, c_2, ... c_m, one valued with respect to x, y for the region within which it is to be considered, and rational with respect to c_1, c_2, ... c_m, where c_1, c_2, ... c_m satisfy $m - n$ algebraic equations $\psi_i(c_1, c_2, \ldots\ldots c_m) = 0$, $(i = 1, \ldots\ldots m - n)$, but are otherwise arbitrary.

$\phi = 0$ defines an n-ply infinite system of curves which satisfy a differential equation of order n, $f(y_n, y_{n-1}, \ldots\ldots y, x) = 0$, the eliminant with respect to c_1, c_2, ... c_m of the $m + 1$ equations

$$\phi = 0,\ \frac{d\phi}{dx} = 0,\ \frac{d^2\phi}{dx^2} = 0, \ldots\ldots, \frac{d^n\phi}{dx^n} = 0,\ \psi_i = 0 \qquad (i = 1, \ldots\ldots m - n).$$

As $f = 0$ depends solely on the manner in which $c_1, c_2, \ldots . c_m$ are involved in $\phi = 0$ and not at all on the character of these quantities, it is obvious that $\phi = 0$ will continue to satisfy this equation when $c_1, c_2, \ldots . c_m$ are replaced by any set of functions of x which satisfy the equations:

$$\sum_1^m \frac{\partial}{\partial c_i}(\phi)\frac{dc_i}{dx} = 0, \quad \sum_1^m \frac{\partial}{\partial c_i}\left(\frac{d\phi}{dx}\right)\frac{dc_i}{dx} = 0, \ldots . \sum_1^m \frac{\partial}{\partial c_i}\left(\frac{d^{n-1}\phi}{dx^{n-1}}\right)\frac{dc_i}{dx} = 0,$$

$$\sum_1^m \frac{\partial}{\partial c_i}(\psi_k)\frac{dc_i}{dx} = 0 \qquad (k = 1, \ldots . m - n)$$

or the single equation

$$\Delta \equiv \begin{vmatrix} \frac{\partial\phi}{\partial c_1}, & \frac{\partial\phi}{\partial c_2}, & \ldots . \frac{\partial\phi}{\partial c_m} \\ \frac{\partial}{\partial c_1}\frac{d\phi}{dx}, & \frac{\partial}{\partial c_2}\frac{d\phi}{dx}, & \ldots . \frac{\partial}{\partial c_m}\frac{d\phi}{dx} \\ \cdots & \cdots & \cdots \\ \frac{\partial}{\partial c_1}\frac{d^{n-1}\phi}{dx^{n-1}}, & \frac{\partial}{\partial c_2}\frac{d^{n-1}\phi}{dx^{n-1}}, & \ldots . \frac{\partial}{\partial c_m}\frac{d^{n-1}\phi}{dx^{n-1}} \\ \frac{\partial\psi_1}{\partial c_1}, & \frac{\partial\psi_1}{\partial c_2}, & \ldots . \frac{\partial\psi_1}{\partial c_m} \\ \cdots & \cdots & \cdots \\ \frac{\partial\psi_{m-n}}{\partial c_1}, & \frac{\partial\psi_{m-n}}{\partial c_2}, & \ldots . \frac{\partial\psi_{m-n}}{\partial c_m} \end{vmatrix} = 0.$$

The eliminant with respect to $c_1, c_2, \ldots . c_m$ of the $m + 1$ equations:

$$\phi = 0, \quad \frac{d\phi}{dx} = 0, \ldots . \frac{d^{n-1}\phi}{dx^{n-1}} = 0, \quad \psi_i = 0, \quad \Delta = 0,$$

a differential equation of order $n - 1$,

$$\chi(y_{n-1}, y_{n-2}, \ldots . y, x) = 0,$$

will in general be a singular solution of the equation $f(y_n, \ldots . y, x) = 0$. Its solution will not be contained in the curve-system which $f = 0$ defines, but it will satisfy this equation.

This is readily proven, as follows. Of the curves $\phi = 0$ there is an $(n - 1)$-

ply infinite system passing through any point x_0, y_0, the parameters of which satisfy the equations

$$\phi(x_0, y_0, c_1, \ldots . c_m) = 0, \quad \psi_i(c_1, c_2, \ldots . c_m) = 0, \qquad (i = 1, \ldots . m - n).$$

Between the parameters $c_1, c_2, \ldots . c_m$ of any curve of this system, and the parameters $c_1 + dc_1, \ldots . c_m + dc_m$ of a second curve of the system consecutive to it, there must hold the relation:

$$\sum_{1}^{m} \frac{\partial\phi(x_0, y_0, c_1, \ldots . c_m)}{\partial c_i} dc_i = 0 \ldots . \qquad (1)$$

For the two curves to touch at x_0, y_0, it is further necessary that

$$\sum_{1}^{m} \frac{d}{dx}\left(\frac{\partial\phi(x, y, c_1, \ldots . c_m)}{\partial c_i}\right)_{x_0, y_0} dc_i = 0 \ldots . \qquad (2)$$

And for their contact to be of the order $n - 1$, still further that

$$\sum_{1}^{m} \frac{d^2}{dx^2}\left(\frac{\partial\phi(x, y, c_1, \ldots . c_m)}{\partial c_i}\right)_{x_0, y_0} dc_i = 0 \ldots .$$

$$\sum_{1}^{m} \frac{d^{n-1}}{dx^{n-1}}\left(\frac{\partial\phi(x, y, c_1, \ldots . c_m)}{\partial c_i}\right)_{x_0, y_0} dc_i = 0 \ldots . (3), \ldots . (n);$$

the common $y_1, y_2, \ldots . y_{n-1}$ of the curves being given by the equations

$$\left(\frac{d\phi}{dx}\right)_{x_0, y_0} = 0, \ldots ., \left(\frac{d^{n-1}\phi}{dx^{n-1}}\right)_{x_0, y_0} = 0.$$

But the eliminant with respect to $dc_1, dc_2, \ldots . dc_m$ of the equations $(1) \ldots . (n)$ and the $m - n$ additional equations

$$\sum_{1}^{m}{}_{i} \frac{\partial\psi_k}{\partial c_i} dc_i = 0$$

is $\Delta = 0$, it having been supposed in the construction of the differential coefficients which constitute the elements of Δ as in the construction of those involved in the present eliminant, that $c_1, c_2, \ldots . c_m$ are independent of x.

It follows immediately that $\chi(y_{n-1}, y_{n-2}, \ldots . y, x) = 0$, the eliminant with respect to $c_1, c_2, \ldots . c_m$ of the equations $\phi = 0$, $\frac{d\phi}{dx} = 0, \ldots . \frac{d^{n-1}\phi}{dx^{n-1}} = 0$,

$\psi_i = 0$, and $\Delta' = 0$, embodies the relation which must exist among the coordinates x, y of any point and the values at that point of the $y_1, y_2, \ldots . y_{n-1}$ of such consecutive curves of the system $\phi = 0$ as there have contact of the order $n - 1$ with each other.

A curve of the system $\chi = 0$ may therefore be described by moving a tracing point from $x_0 y_0$ to $x_1 y_1$ along an infinitesimal arc of $\phi_0 = 0$, a curve of the system $\phi = 0$; thence to $x_2 y_2$ along an infinitesimal arc of the curve $\phi_1 = 0$ which has a contact of the order $n - 1$ with $\phi_0 = 0$ at $x_0 y_0$; thence again to $x_3 y_3$ along an infinitesimal arc of $\phi_2 = 0$ which has contact of the order $n - 1$ with $\phi_1 = 0$ at $x_1 y_1$, etc. This curve, since it has the same $y_1, y_2, \ldots . y_{n-1}$ with $\phi_1 = 0$ at $x_0 y_0$ and also at the consecutive point $x_1 y_1$, has contact of the n^{th} order with $\phi_1 = 0$ at $x_0 y_0$; in like manner contact of the n^{th} order with $\phi_2 = 0$ at $x_1 y_1$, etc. It therefore satisfies the equation $f(y_n, y_{n-1}, \ldots . y, x) = 0$. Therefore

The equation $\chi = 0$ *contains the envelope system of* $\phi = 0$, *a system of curves each of which has at each of its points contact of the* n^{th} *order with a curve of* $\phi = 0$; *and this system satisfies the equation* $f = 0$.

It should be noticed, however, that $\chi = 0$ may contain systems of curves besides this envelope system—which do not satisfy the equation $f = 0$; may indeed contain such curves only.

By the hypothesis made at the outset there is a $1 - 1$ correspondence between the curves of $\phi = 0$ and the systems of values of the parameters $c_1, c_2, \ldots . c_m$. The equations $\phi = 0$, $\dfrac{d\phi}{dx} = 0, \ldots . \dfrac{d^{n-1}\phi}{dx^{n-1}} = 0$, $\psi_i = 0$, give $c_1, c_2, \ldots . c_m$ in terms of $x, y, y_1, \ldots . y_{n-1}$ and therefore determine the sets of values of these parameters which belong to that set of the curves $\phi = 0$ which pass through any point x, y, and there have given $y_1, y_2, \ldots . y_{n-1}$. In the general case these sets of values are distinct from one another, and so the curves of the set are distinct curves. But it may happen that two or more of the sets of values are the same. The corresponding curves then cannot be distinct, but must be but different branches of one and the same curve.

For the occurrence of loci of such singular points more conditions must be satisfied than there are independent parameters; ϕ must have a special form. But when they do occur, they satisfy the equation $\chi = 0$.

This may be readily seen by regarding $c_1, c_2, \ldots . c_m$ as the coordinates of a point in a space of m dimensions S_m.

$\phi = 0$, $\frac{d\phi}{dx} = 0$, ... $\frac{d^{n-1}\phi}{dx^{n-1}} = 0$, $\psi_i = 0$ are then the equations of m forms in this space, each of dimensionality $m - 1$, which have in common a finite number of points, generally distinct. If, however, two (or more) of these points coincide, the corresponding tangent S_{m-1}'s of the m forms have a line (S_1) in common, and the determinant Δ—whose elements are the coefficients of these tangent S_{m-1}'s—vanishes; that is, the condition $\chi = 0$ is satisfied.

Besides the envelope system, therefore, $\chi = 0$ *includes the loci of points at which curves of the system* $\phi = 0$ *have the singularity of two curve branches having contact of the* $n - 1^{th}$ *order with each other, when such loci exist.*

These loci, however, do not satisfy the equation $f = 0$, and hence constitute no part of its singular solution; for while $\chi = 0$ for the singular point itself on a curve $\phi_0 = 0$, this equation will generally not be satisfied for consecutive points of the curve, these points not being singular; or the values of y_n for $\chi = 0$ and $\phi = 0$ at the singular point are different.

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